Midterm 2 Duration: 45 minutes

*This test has **5 questions** on **7 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Signature: ____

Question:	1	2	3	4	5	Total
Points:	6	15	7	7	5	40
Score:						

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 2 to 3 marks, and not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

2 marks 1. (a) Let $y = 10^{1-x^2}$. Find $\frac{dy}{dx}$.

Answer: $\frac{dy}{dx} = -2\ln(10)x \cdot 10^{1-x^2}$

Solution: Use the chain rule

$$\frac{dy}{dx} = \ln(10) \cdot 10^{1-x^2} \cdot (-2x)$$

2 marks (b) Evaluate $\tan^{-1}\left(\tan\left(\frac{11\pi}{4}\right)\right)$.

Answer: $-\frac{\pi}{4}$

Solution: As done in class, one way to solve this problem is to draw a unit circle and mark down the angle of $\frac{11\pi}{4}$ in the second quadrant. Shifting the angle back into the range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ yields $-\frac{\pi}{4}$ because $\tan x$ is negative in the second and fourth quadrant.

2 marks (c) $y = (\sin x)^{\ln x}$. Find $\frac{dy}{dx}$.

Answer: $y' = \left(\frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}\right) y$

Solution: Use logarithmic differentiation.

$$\ln y = \ln x \cdot \ln(\sin x)$$
$$\frac{1}{y}y' = \frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \frac{1}{\sin x} \cdot \cos x$$

Multiplying through by y gives the answer in the box.

3 marks 2. (a) Simplify $\sin(\sec^{-1} x)$. Note: $\sec^{-1} x$ is the inverse function of $\sec x$.

Answer: $\frac{\sqrt{x^2-1}}{x}$

Solution: Let $y = \sec^{-1} x$ then $\sec y = x$ or equivalently $\cos y = \frac{1}{x}$. Drawing a triangle with angle y as done in class gives $\sin y = \frac{\sqrt{x^2-1}}{x}$.

3 marks (b) Find all x-coordinates at which the tangent line to the curve $x^2y^2 = 16$ has slope -1.

Answer: $x = \pm 2$

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Solution: Implicit differentiation gives

$$2xy^2 + x^2 \cdot 2yy' = 0.$$

Plugging in $y' = -1$ and factoring gives

$$2xy(y - x) = 0.$$

This implies $x = y$ since there are no points on the curve with $x = 0$ or $y = 0$. The curve's equation thus becomes $x^4 = 16$ or $x = \pm 2$.

3 marks (c) Which of the following functions satisfies $\dfrac{\mathrm{d}p}{\mathrm{d}t} = p^3$?

A) $p(t) = \frac{1}{4}t^4$

B) $p(t) = -\frac{1}{t+3}$

C) $p(t) = \dfrac{1}{\sqrt{1-2t}}$

D) None of these.

Answer: C)

Solution: By differentiating each expression for $p(t)$ separately, we conclude that C) is the correct answer, i.e,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{2(1-2t)^{3/2}} \cdot (-2) = \frac{1}{(1-2t)^{3/2}} = \left(\frac{1}{\sqrt{1-2t}}\right)^3 = p^3$$

3 marks

(d) Find the absolute maximum and absolute minimum values of $y = x + \frac{1}{x}$ on $[0.2, 4]$.

Answer: max. value is $y(0.2) = 5.2$, min. value is $y(1) = 2$

Solution: Use the closed interval method. Start by finding critical numbers by checking where the derivative equals zero or isn't defined:

$$y' = 1 - \frac{1}{x^2} = 0$$

So $x = 0, \pm 1$. Ignore 0 and -1 since they aren't in $[0.2, 4]$. We compare

$$\begin{aligned} y(0.2) &= 0.2 + 5 \\ y(1) &= 2 \\ y(4) &= 4 + 0.25 \end{aligned}$$

to get the answer in the box.

3 marks

(e) Write down the degree 2 Taylor polynomial of $f(x) = \tan x$ at $x = \pi/4$.

Answer: $T_2(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2$

Solution: We need to determine

$$T_2(x) = \sum_{k=0}^{2} c_k (x - \pi/2)^k$$

where $c_k = \frac{f^{(k)}(\pi/2)}{k!}$. We have

$$f^{(0)}(x) = \tan x \quad f'(x) = \frac{1}{\cos^2 x} \quad f''(x) = 2\frac{\tan x}{\cos^2 x}$$

and so

$$c_0 = 1 \qquad c_1 = 2 \qquad c_2 = 2$$

which gives the answer in the box.

Full-Solution Problems. In questions 3–5, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

3 marks

3. (a) Use a suitable linear approximation to estimate the value of $\sqrt[3]{26^2}$. Express your answer as a single fraction.

Answer: $\frac{79}{9}$

Solution: Let $f(x) = x^{2/3}$ and expand around $x = 27$. Then

$$\begin{aligned} f(27) &= 9 \\ f'(x) &= \frac{2}{3}x^{-1/3} \\ f'(27) &= \frac{2}{9} \\ T_1(x) &= f(27) + f'(27)\cdot(x-27) \\ &= 9 + \frac{2}{9}(x-27) = \frac{2}{9}x + 3 \end{aligned}$$

and thus

$$T_1(26) = 9 + \frac{2}{9}\cdot(-1) = \frac{79}{9}.$$

4 marks

(b) Show that the upper bound for the error in this approximation is less than $-\dfrac{1}{1000}$.

Solution: The error is given by

$$E_1 = \frac{1}{2!}f''(c)(26-27)^2 = \frac{1}{2}f''(c)$$

for some $26 \le c \le 27$. Now $f''(x) = -\frac{2}{9}x^{-4/3}$ has a negative power of x which is decreasing but then the minus sign outside makes $f''(x)$ increasing. Thus the right endpoint of the interval $[26, 27]$ makes $f''(x)$ largest and we obtain

$$\begin{aligned} f''(x) = -\frac{2}{9}x^{-4/3} &\le -\frac{2}{9}27^{-4/3} \qquad \text{for } 26 \le x \le 27 \\ &= -\frac{2}{9}\cdot\frac{1}{3^4} = -\frac{2}{9\cdot 81} \end{aligned}$$

So the error is bounded by

$$E_1 \le -\frac{1}{2}\cdot\frac{2}{9\cdot 81} = -\frac{1}{729} < -\frac{1}{1000}$$

as required.

4. A tanker off the coast of B.C. breaks in half spilling 10^4 m^3 of oil all at once. The oil floats on the surface and spreads. It has the shape of a cylinder whose height is becoming thinner and thinner. The height, h, of the oil slick is uniform and given by

$$h(t) = \frac{1}{10^4 t}$$

where t is time in hours.

2 marks (a) Find the radius of the slick 4 hours later.

Solution: Let t be time measured in hours, r be the radius of the cylinder, and V be the volume of the cylinder. Then

$$V = \pi r(t)^2 h(t).$$

Thus

$$r(t) = \sqrt{\frac{10^4 \cdot 10^4 t}{\pi}} \qquad \text{and} \qquad r(4) = 10^4 \frac{2}{\sqrt{\pi}} = \frac{20,000}{\sqrt{\pi}}.$$

5 marks (b) Find how quickly the slick is spreading across the water surface 4 hours later.

Solution: The quick way is to just differentiate the above expression for $r(t)$ to get

$$r'(t) = \frac{10^4}{2\sqrt{\pi}} t^{-1/2} \qquad \text{so} \qquad r'(4) = \frac{10^4}{4\sqrt{\pi}} = \frac{2,500}{\sqrt{\pi}}.$$

The long (and tedious) way is to differentiate the volume formula implicitely (treating V as a constant):

$$0 = \pi(2rr'h + r^2h') \qquad (*)$$

We need

$$h'(t) = -\frac{1}{10^4 t^2} \qquad \text{so that} \qquad h'(4) = -\frac{1}{10^4 \cdot 16}.$$

Plugging $t = 4$ into equation $(*)$ and dividing by π we get

$$0 = 2 \cdot 10^4 \frac{2}{\sqrt{\pi}} \cdot r' \cdot \frac{1}{10^4 \cdot 4} + \frac{4}{\pi} \cdot 10^8 \cdot \left(-\frac{1}{10^4 \cdot 16}\right) = \frac{r'}{\sqrt{\pi}} - \frac{10^4}{4\pi}$$

and solving for r':

$$\frac{r'}{\sqrt{\pi}} = \frac{10^4}{4\pi}$$

$$r' = \frac{10^4\sqrt{\pi}}{4\pi} = \frac{10^4}{4\sqrt{\pi}} = \frac{2,500}{\sqrt{\pi}}.$$

5 marks

5. Naobi is boiling water on the stove to make a cup of coffee. The heating element is at 200°C and the water in her pot is initially at 20°C room temperature. After one minute her water has reached 40°C. How long will it take for her water to reach the boiling point? Your answer may be left in "calculator-ready" form.

Solution: Let T be the temperature of the can in degree Celsius and t be the time in minutes.

We need to fit the curve $T(t) = Ae^{kt} + E$ to the given data where E is the ambient temperature. In this case the ambient temperature is that of the heating element, i.e., 200°C. $A = T(0) - E = 20 - 200 = -180$. The other data point is $t = 1$ with $T(1) = 40$. Plugging this into the equation we have thus far we get

$$40 = -180e^{k} + 200.$$

Solving for k yields $k = \ln \frac{8}{9}$. Thus

$$T(t) = -180e^{\ln\left(\frac{8}{9}\right)t} + 200.$$

Solving

$$100 = -180e^{\ln\left(\frac{8}{9}\right)t} + 200$$

for t we find $t = \dfrac{\ln \frac{5}{9}}{k} \approx 5$ min.